Trigonometric Determination of the Optimal Tilt Angle of Photovoltaic Panels: Analytical Generalisation and Clear-Sky Quantification for the Fergana Valley, Uzbekistan

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Abstract. *Fixed-tilt photovoltaic (PV) arrays dominate the fast-growing distributed solar segment of Uzbekistan, and a widely reproduced shortcut states that the optimal tilt of a south-facing panel is the complement of the solar elevation, $\beta = 90° - \alpha$. This study shows that the rule is the solar-meridian limit of the general incidence relation and derives its exact replacement. The angle of incidence was optimised analytically with respect to tilt under a fixed southward azimuth and embedded in a minute-resolution, whole-year clear-sky simulation for Fergana (40.386° N, 71.786° E, 580 m) combining the Cooper declination relation, the Spencer equation of time, the Hottel beam-transmittance model, the Liu-Jordan diffuse correlation and isotropic transposition. Applied hour by hour on the equinox the complement rule predicts normal incidence while the true incidence angle reaches 63.3°, a cosine loss of 55 %. The exact optimum equals $\varphi - \delta$ at solar noon and varies by only 5.5° across an equinox day; consequently continuous tilt tracking yields +1.4 % over a once-daily tilt reset and +7.2 % over the annual-optimal fixed tilt of 36.0°, whereas releasing the azimuth constraint yields +20.9 % (horizontal north-south axis) and +39.3 % (dual axis). The ceiling on any tilt-only mechanism is $\cos\theta^{*} = \sqrt{(1 - \cos^2\alpha \sin^2\gamma_s)}$, an irreducible azimuth penalty. A monthly tilt schedule for the Fergana Valley and a two-panel validation protocol are provided.*

Keywords: *solar geometry; angle of incidence; optimal tilt angle; trigonometric modelling; photovoltaic yield; single-axis tracking; Fergana Valley.*

1. Introduction

Photovoltaic conversion has become a structural component of the Central Asian power system. By the end of 2025 Uzbekistan operated 148 power plants totalling 25,797 MW, of which 3,930 MW was utility-scale PV, while the national Solar House net-metering scheme added 2,508 MW of largely rooftop capacity between April 2023 and August 2026 against a 2030 target of 25 GW of solar and wind [11, 12]. The distributed segment is therefore of the same order as the utility fleet and consists overwhelmingly of small arrays on fixed frames whose tilt is chosen once by non-specialists. For these systems orientation is one of the few design variables that costs nothing, since plane-of-array irradiance is governed to first order by the cosine of the angle of incidence.

The problem is classical: Duffie and Beckman [1] give the geometric framework, Kalogirou [2] and Masters [3] its engineering consequences, Reda and Andreas [4] a high-accuracy position algorithm, and Yadav and Chandel [5] review decades of tilt optimisation. Regionally, Nosirov et al. [6] obtained an optimal fixed tilt of 33° for Parkent (41.3° N) and reported up to about 4 % from two- and four-season schedules.

Two gaps motivate this work. The first is methodological. Much applied and didactic material

in the region introduces the daily optimum through

$$\beta_{opt} = 90^\circ - \alpha \quad (1)$$

derived by requiring the Sun, the surface normal and the vertical to be coplanar. That holds only in the local meridian plane. Away from solar noon the Sun acquires an azimuth component that no rotation about a horizontal east-west axis can compensate, yet equation (1) is routinely applied hour by hour, producing schedules that predict normal incidence when the true incidence angle exceeds 60°. The error inverts the qualitative conclusion about what a tilt mechanism is worth. The second gap is quantitative: tilt adjustment at a permanently southward azimuth is seldom separated from true single-axis tracking, although the two differ by a factor of three in delivered energy.

The objectives are therefore to derive the exact closed-form tilt optimum of a plane constrained to a southward azimuth and to identify equation (1) as its meridian-plane limit; to quantify the error of applying equation (1) outside that limit; to compute the annual yield of eight orientation strategies for the Fergana Valley in a reproducible clear-sky simulation; and to issue a practical monthly schedule with an experimental validation protocol.

2. Materials and Methods

Site and radiation model. All computations refer to Fergana city: latitude φ = 40.3864° N, longitude 71.7864° E, elevation 580 m, standard meridian 75° E (UTC+5), ground reflectance ρ_g = 0.20. Solar declination was obtained from the Cooper relation δ = 23.45° sin[360°(284 + n)/365] and the mean-to-apparent time offset from the Spencer series; the hour angle follows as ω = 15°(t_s − 12), negative before noon. Solar elevation α and azimuth γ_s, measured from due south and positive westward, are [1]

$$\sin\alpha = \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos\omega \quad (2)$$

$$\cos\gamma_s = \frac{\sin\alpha\sin\varphi - \sin\delta}{\cos\alpha\cos\varphi}, \quad \operatorname{sign}(\gamma_s) = \operatorname{sign}(\omega) \quad (3)$$

These reproduce elevation to better than 0.2° against Reda and Andreas [4], an order of magnitude finer than the tilt resolution used here. Clear-sky beam transmittance was computed with the Hottel model [7] for a 23 km visibility atmosphere with mid-latitude-summer correction factors (r_0 = 0.97, r_1 = 0.99, r_k = 1.02), diffuse transmittance from the Liu-Jordan correlation τ_d = 0.271 − 0.294 τ_b [8], and extraterrestrial normal irradiance from G_{on} = 1367[1 + 0.033 cos(360°n/365)] W/m², giving beam normal irradiance $G_b = G_{on}\tau_b$ and horizontal diffuse $G_d = G_{on}\tau_d \sin\alpha$.

Angle of incidence. For a surface of tilt β and azimuth γ, the scalar product of the outward normal and the solar unit vector gives

$$\cos\theta = \cos\alpha\sin\beta\cos(\gamma_s - \gamma) + \sin\alpha\cos\beta \quad (4)$$

equivalent to the expanded form of Duffie and Beckman [1, eq. 1.6.2]. Negative values denote rear-surface illumination and were set to zero. For the south-facing case γ = 0,

$$\cos\theta = \cos\alpha\cos\gamma_s\sin\beta + \sin\alpha\cos\beta \quad (5)$$

Only at meridian crossing, γ_s = 0, does cos γ_s equal unity and equation (5) collapse to $\theta = |\alpha + \beta - 90^\circ|$. The complement rule of equation (1) is thus the ω = 0 limit of equation (5); the factor cos γ_s is precisely the term omitted in the meridian-plane treatment, and it vanishes as the Sun approaches the prime vertical.

Analytical tilt optimum. Setting $d(\cos\theta)/d\beta = 0$ in equation (5) gives

$$\tan\beta^* = \frac{\cos\alpha\cos\gamma_s}{\sin\alpha} \quad (6)$$

The second derivative equals −cos θ and is negative wherever the surface is illuminated, so the stationary point is a maximum. Substituting the identity cos α cos γ_s = sin φ cos δ cos ω − cos φ sin δ yields the optimum in terms of site and time,

$$\tan\beta^* = \frac{\sin\varphi\cos\delta\cos\omega - \cos\varphi\sin\delta}{\sin\varphi\sin\delta + \cos\varphi\cos\delta\cos\omega} \quad (7)$$

which at ω = 0 becomes $\tan(\varphi - \delta)$, that is

$$\beta^*(\omega = 0) = \varphi - \delta \quad (8)$$

the exact noon optimum, equivalent to equation (1) because $\alpha(\omega = 0) = 90^\circ - \varphi + \delta$. Equation (7) therefore supersedes equation (1) while containing it as a special case. Back-substitution of equation (6) into equation (5) gives the greatest projection attainable by any tilt at a fixed southward azimuth,

$$\cos\theta^* = \sqrt{\sin^2\alpha + \cos^2\alpha\cos^2\gamma_s} = \sqrt{1 - \cos^2\alpha\sin^2\gamma_s} \quad (9)$$

Equation (9) is the central structural result: the performance of an ideal tilt-only mechanism is limited

by an azimuth penalty that no rotation about a horizontal east-west axis can reduce. It is small near solar noon and near the winter solstice, but dominant in the early and late hours of summer days when the Sun rises well north of east.

Transposition, conversion and implementation. Plane-of-array irradiance followed from the isotropic Liu-Jordan sky model,

$$G_T = G_b \max(\cos\theta, 0) + G_d \frac{1+\cos\beta}{2} + \rho_g (G_b \sin\alpha + G_d) \frac{1-} \quad (10)$$

integrated over the illuminated part of each day as a Riemann sum on a one-minute grid. The isotropic model is conservative with respect to the hypothesis under test, since anisotropic formulations concentrate diffuse radiation around the solar disc and would inflate the computed advantage of tracking. Electrical output was checked with a nominal-operating-cell-temperature model, $T_c = T_a + (NOCT - 20)G_T/800$ with NOCT = 45 °C and $P = P_{STC}(G_T/1000)[1 + \gamma_P(T_c - 25)]$ with $\gamma_P = -0.40$ %/K, sweeping ambient temperature parametrically over 10-40 °C rather than imposing a climatological series.

The model was implemented in Python 3 with NumPy and evaluated at one-minute resolution for all 365 days (525,600 states per configuration). Annual and monthly tilt optima were located by exhaustive scan over 0-70° at 0.5° resolution rather than gradient search. Eight configurations were simulated on identical time and radiation inputs so that differences are geometric only: C1 horizontal; C2 fixed annual-optimal tilt; C3 monthly tilt reset; C4 daily reset at $\beta = \varphi - \delta$; C5 five manual positions per day in solar-time blocks bounded at 09, 11, 13 and 15 h; C6 continuous tilt-only tracking by equation (6), clipped to [0, 90°]; C7 horizontal north-south axis tracker, $\cos\theta = \sqrt{(\cos^2\theta_z + \cos^2\delta \sin^2\omega)}$; and C8 ideal dual-axis tracking. C1-C6 share a permanently southward azimuth. Three checks were performed: equation (6) coincided with the numerical maximum of equation (5) at every time step; equation (8) reproduced the scanned noon optimum exactly on all cardinal days; and reducing the time step to fifteen seconds changed annual totals by less than 0.02 %.

3. Results

Failure of the meridian-plane approximation. Table 1 inserts the tilt prescribed by equation (1) into the exact relation of equation (5) on the March equinox. The rule predicts $\theta = 0$ at every hour by construction, whereas the true incidence angle rises to 31.0° two hours from noon and 63.3° four hours from noon. The discrepancy is a direct expression of equation (9): because the azimuth term is absent from the meridian-plane derivation, the approximation attributes to tilt an authority it does not possess.

Table 1. Incidence angle actually obtained when $\beta = 90° - \alpha$ is applied hour by hour. Fergana, 20 March.

Solar time	α (°)	γ_s (°)	β applied (°)	θ predicted	θ exact	cos θ	Cosine loss
08:00	21.8	−68.9	68.2	0.0°	63.3°	0.449	55.1 %
10:00	40.6	−41.2	49.4	0.0°	31.0°	0.857	14.3 %
12:00	48.8	0.0	41.2	0.0°	0.0°	1.000	0.0 %
14:00	40.6	+41.2	49.4	0.0°	31.0°	0.857	14.3 %
16:00	21.8	+68.9	68.2	0.0°	63.3°	0.449	55.1 %

Near-invariance of the optimal tilt. The exact optimum of equation (6) is almost stationary within a day. With the Sun above 5° elevation it spans 41.2-46.7° on the equinox, 0.0-16.9° at the June solstice and 63.8-81.6° in December, in each case anchored at its noon value $\varphi - \delta$. The reason is visible in equation (7): numerator and denominator vary with ω through the same cos ω factor, so their ratio is far less time-sensitive than either term. Consequently a fixed plane at 36.0° never loses more than 0.004 in cosine relative to the ideal hour-by-hour optimum anywhere on the equinox day. The optimal tilt is effectively a function of declination alone, and declination changes by at most 0.4° per day: a tilt mechanism is a seasonal device in the costume of a diurnal one.

Optimal fixed and monthly tilts. Exhaustive scanning identified an annual optimum of $\beta_{opt} = 36.0°$, that is $\varphi - 4.4°$. The optimum is flat: every tilt between 30.0 and 42.0° lies within 0.5 % of the maximum and the 1 % band extends from 27.5 to 44.5°, so

installation tolerance is not a practical concern. The monthly schedule (Table 2) spans 60.5°, closely tracking the 46.9° declination swing augmented by the diffuse and albedo terms of equation (10). Its benefit is concentrated in the summer and winter months, where resetting gains 11-14 %; at the equinoxes the annual compromise is already near-optimal and the gain falls below 1 %.

Annual yield of the eight configurations. Table 3 gives the principal quantitative result: clear-sky plane-of-array irradiation integrated over the full year, with the fixed annual-optimal configuration as reference.

Table 2. Monthly optimal tilt for a south-facing plane, Fergana Valley (°).

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
βopt	64.5	57.0	44.0	27.5	13.0	6.0	9.5	22.0	38.0	53.0	63.0	66.5

Table 3. Annual clear-sky plane-of-array irradiation by orientation strategy, Fergana (40.386° N). Reference: C2, fixed south-facing tilt of 36.0°.

Code	Configuration	MJ/m² yr	kWh/m² yr	vs C2	Azimuth freedom
C1	Horizontal	6817	1894	−15.8 %	none
C2	Fixed 36.0° (reference)	8097	2249	0.0 %	none
C3	Monthly tilt reset	8588	2385	+6.1 %	none
C4	Daily reset, $\beta = \varphi - \delta$	8560	2378	+5.7 %	none
C5	Five manual positions per day	8661	2406	+7.0 %	none
C6	Continuous tilt-only tracking	8677	2410	+7.2 %	none
C7	Horizontal N-S axis tracker	9785	2718	+20.9 %	continuous
C8	Ideal dual-axis tracking	11278	3133	+39.3 %	continuous

Three findings follow. First, correct tilt selection alone is worth 18.8 % relative to a horizontal plane, which remains the largest zero-cost gain available to a fixed installation. Second, the entire family of tilt-only strategies C3-C6 is confined to +5.7 % to +7.2 %: a once-daily reset captures 80 % of the continuous-tracking gain, a monthly reset 85 % and five manual positions per day 97 %, so continuous tracking exceeds a daily reset by only +1.4 %. Notably the noon-exact rule C4 yields slightly less than the twelve-step schedule C3, because it optimises the single instant of highest irradiance whereas the monthly optimum maximises the integral. Third, releasing the azimuth constraint changes the picture entirely: +20.9 % for a horizontal north-south axis and +39.3 % for dual-axis tracking, five times the maximum obtainable from tilt alone. The seasonal structure is strong: on the June solstice C7 gains 52.8 % over C2 but on the December solstice it loses 19.5 %, since a horizontal rotation axis cannot be tilted towards a Sun that never rises above 26°. Repeating the comparison in electrical terms, specific yield falls from 2188 to 1918 kWh/kWp as ambient temperature rises from 10 to 40 °C, yet the relative advantage of tilt tracking stays within 0.04 percentage points of 6.9 %, so the geometric ranking transfers to electrical output unchanged.

4. Discussion

The results admit one compact explanation. Equation (9) separates the attainable projection into an elevation term and an azimuth penalty; tilt rotation nulls the first but is powerless against the second, which is largest exactly when days are long and the potential gain greatest. At 08:00 on the June solstice at Fergana the Sun stands north of east, cos γ_s is negative, equation (6) returns a negative tangent, and the physical optimum is the clipped value $\beta^* = 0$: the best a south-facing plane can do is lie flat. Combined with the near-invariance of β^* within a day, this establishes the correct design hierarchy: choose the tilt well (18.8 %), reset it seasonally if convenient (6.1 %), and invest in azimuth freedom only if a further 14-32 percentage points can be financed.

For the 2.5 GW installed under the Solar House scheme the actionable finding is the first line of that

hierarchy, not the last. Setting a frame to 36° rather than laying modules flat is worth 18.8 % at no capital cost, more than twice the entire gain available from any tilt-tracking mechanism. Where the mounting permits adjustment the schedule of Table 2 adds 6.1 %; an explicitly optimised two-position approximation recovers four fifths of that, 15° from April to August and 54° from September to March giving +4.9 % at two adjustments per year. Conversely the case for motorised tilt-only tracking on small systems is weak, since 7.2 % must finance actuator, controller, structural reinforcement for wind loading on a moving frame, parasitic consumption and maintenance against a module cost that has fallen far faster than mechanical cost. One caution applies to the steep winter settings: 66.5° presents a large sail area to the valley's winter winds and increases inter-row shading for a given row pitch, neither of which the present geometric model captures.

The computed optimum of 36.0° for Fergana is consistent with the 33° obtained independently for Parkent at 41.3° N by Nosirov et al. [6], the difference attributable to their use of measured data in which winter cloudiness devalues the low-Sun months and shifts the optimum shallower. The seasonal-adjustment gain of about 6 % likewise sits within the 4-8 % band reported for mid-latitude sites by Yadav and Chandel [5], at the upper end as the clear-sky assumption implies. What this study adds is the explicit identification of equation (1) as a meridian-plane limit, the closed-form ceiling of equation (9), and the separation of tilt-only adjustment from true single-axis tracking on a common radiative basis. The derivation is also an unusually clean piece of applied trigonometry: a scalar product gives equation (4), one differentiation gives equation (6), one substitution gives equation (7), a limit recovers the schoolbook rule, and back-substitution gives a physically interpretable bound. Table 1 illustrates something mathematics teaching rarely has occasion to show, that a correct derivation under an unstated assumption is more dangerous than none, because it carries the authority of a formula.

Limitations. The radiation model is cloudless: the computed horizontal annual total of 1894 kWh/m² exceeds measured global horizontal irradiation for the region by roughly 15-20 %, so absolute yields are clear-sky upper bounds. The ratios that carry the argument are far more robust than the levels, although cloud would compress all gains in Table 3 towards zero, strengthening rather than weakening the central conclusion. The isotropic sky model neglects circumsolar and horizon brightening, which would raise the tracking configurations. Soiling is omitted and interacts with tilt, since steeper planes shed dust and precipitation more readily. Horizon obstruction is neglected, which in an intermontane basin truncates the first and last hours of the day and so reduces the azimuth-tracking advantage of C7 and C8. Inter-row shading, inverter clipping and mismatch lie outside a single-plane formulation.

Validation protocol. The model admits direct testing. Two modules of identical make, rating and age are mounted side by side on one structure with a common southward azimuth and no mutual or external shading; one is fixed at 36.0°, the other reset monthly according to Table 2. Current, voltage, plane-of-array irradiance and back-of-module temperature are logged at both planes at intervals of one minute or less under a common maximum-power-point load, with power from P = UI and daily energy from its time integral. The quantity to compare with Table 3 is the ratio of daily energies rather than their absolute values, since the ratio cancels most calibration and soiling error. Measurement should span a full month per season, separate clear from overcast days by a measured clearness index, treat the daily ratios by a paired-sample test, and periodically exchange the modules between positions to control for unit-to-unit variation.

5. Conclusions

1. The complement rule $\beta = 90° - \alpha$ is the solar-meridian limit ($\omega = 0$) of the general incidence relation and is invalid at other hour angles; applied hour by hour on the equinox it predicts normal incidence while the true incidence angle reaches 63.3°, a cosine loss of 55 %.

2. The exact tilt optimum of a plane at fixed southward azimuth is $\tan \beta^* = \cos \alpha \cos \gamma_s / \sin \alpha$, equivalently equation (7) in φ, δ and ω, reducing to $\beta^* = \varphi - \delta$ at solar noon; the maximum attainable projection is $\cos \theta^* = \sqrt{(1 - \cos^2\alpha \sin^2\gamma_s)}$, an azimuth penalty no tilt rotation can remove.

3. Because β^* varies by only 5.5° across an equinox day, continuous tilt tracking adds just +1.4 % over a once-daily reset and +7.2 % over a fixed annual-optimal tilt, while a monthly reset captures +6.1 %;

motorised tilt-only tracking is therefore hard to justify on small systems.

4. For Fergana the annual-optimal fixed tilt is 36.0°, worth +18.8 % against a horizontal plane, with every setting between 30 and 42° within 0.5 % of the maximum; a twelve-value monthly schedule is given in Table 2.

5. Azimuth freedom, not tilt precision, holds the remaining energy: +20.9 % for a horizontal north-south axis tracker and +39.3 % for ideal dual-axis tracking under clear skies. These rankings are invariant to module temperature over a 30 K ambient range and so transfer from irradiation to electrical yield without modification.

The analytical results are site-independent; the numerical schedule applies to the Fergana Valley and can be regenerated for any latitude from equations (2)-(10).

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